MATHEMATICS

1. Find number of common terms in the two given series

4, 9, 14, 19……. up to 25 terms and

3, 9, 15, 21 …..up to 37 terms

(1) 4 (2) 7 (3) 5 (4) 3

Ans. **(1)**

Sol. 4, 9, 14, 19, ……. 124 $\rightarrow d_1 = 5$

3, 9, 15, 21 …….. 219 $\rightarrow d_2 = 6$

1^{st} common term = 9 and common difference of common terms = 30

Common terms are 9, 39, 69, 99

4 common terms

2. Let $8 = 3 + \frac{3+p}{4} + \frac{3+2p}{4^2} + \ldots \infty$, then p is

(1) 9 (2) $\frac{5}{4}$ (3) 3 (4) 1

Ans. **(1)**

Sol. $8 = 3 + \frac{3+p}{4} + \frac{3+2p}{4^2} + \ldots\ldots(i)$

multiply both sides by $\frac{1}{4}$, we get

$2 = \frac{3}{4} + \frac{3+p}{4^2} + \ldots\ldots (ii)$

Equation (i) – equation (ii)

$\Rightarrow 6 = 3 + \frac{p}{4} + \frac{p}{4^2} + \ldots\ldots$

$\Rightarrow 3 = \frac{p}{4\left(1-\frac{1}{4}\right)} \Rightarrow p = 9$

3. For $\frac{x^2}{25} + \frac{y^2}{16} = 1$, find the length of chord whose mid point is $P\left(1, \frac{2}{5}\right)$

(1) $\frac{\sqrt{1681}}{5}$ (2) $\frac{\sqrt{1481}}{5}$ (3) $\frac{\sqrt{1781}}{5}$ (4) $\frac{\sqrt{1691}}{5}$

Ans. **(4)**

Sol. By $T = S_1$

$$\Rightarrow \frac{x}{25}+\frac{y}{16}=\frac{1}{25}+\frac{4}{25}\cdot\frac{1}{16}$$

$$\Rightarrow \frac{x}{25}+\frac{y}{40}=\frac{4+1}{100}$$

$$\Rightarrow \frac{x}{25}+\frac{y}{40}=\frac{1}{20}$$

$\Rightarrow 8x + 5y = 10$

$$\Rightarrow \frac{x^2}{25}+\left(\frac{10-8x}{5}\right)^2\frac{1}{16}=1$$

$$\Rightarrow \frac{x^2}{25}+\frac{4}{25}\left(\frac{5-4x}{16}\right)^2=1$$

$$\Rightarrow x^2+\frac{(5-4x)^2}{4}=25$$

$\Rightarrow 4x^2 + (5 - 4x)^2 = 100$

$\Rightarrow 20x^2 - 8x - 15 = 0$

$x_1 + x_2 = 2$

$$x_1x_2 = \frac{-15}{4}$$

length of chord $= |x_1 - x_2|\sqrt{1+m^2}$

$$= \frac{\sqrt{1691}}{5}$$

4. If $f(x) = x^3 + x^2f'(1) + xf''(2) + f'''(3)$, then find $f'(10)$.

Ans. **(202)**

Sol. $f'(x) = 3x^2 + 2xf'(1) + f'(2)$

$f''(x) = 6x + 2f'(1)$

$f'''(3) = 6$

$f'(1) = -5$

$f''(2) = 2$

$\Rightarrow f'(10) = 300 + 20(-5) + 2$

$= 202$

5. Let $\int_0^1 \frac{dx}{\sqrt{x+3}+\sqrt{x+1}} = A + B\sqrt{2} + C\sqrt{3}$ then the value of $2A + 3B + C$ is

(1) 3 (2) 4 (3) 5 (4) 6

Ans. **(1)**

Sol. On rationalising

$$\int_0^1 \frac{(\sqrt{x+3}-\sqrt{x+1})}{2}dx$$

$$= \frac{2}{3.2}\left\{(x+3)^{3/2}-(x+1)^{3/2}\right\}_0^1$$

$$= \frac{1}{3}\{8-3\sqrt{3}-(2\sqrt{2}-1)\}$$

$$= \frac{1}{3}\{9-3\sqrt{3}-2\sqrt{2}\}$$

$$= \left(3-\sqrt{3}-\frac{2\sqrt{2}}{3}\right) : A = 3, B = -\frac{2}{3}, C = -1$$

$\therefore$ 2A + 3B + C = 6 – 2 – 1 = 3

6. If |z – i| = |z – 1| = |z + i| , z $\in$ C, then the numbers of z satisfying the equation are

(1) 0 (2) 1 (3) 2 (4) 4

Ans. **(2)**

Sol. z is equidistant from 1, i, & – i

only z = 0 is possible

$\therefore$ number of z equal to 1

7. If sum of coefficients in $(1-3x+10x^2)^n$ and $(1+x^2)^n$ is A and B respectively then

(1) $A^3 = B$ (2) $A = B^3$ (3) A = 2B (4) A = B

Ans. **(2)**

Sol. $A = 8^n$ $B = 2^n$

(B) $\therefore$ $A = B^3$

8. Let $a_1, a_2, \ldots, a_{10}$ are 10 observations such that $\sum_{i=1}^{10} a_i = 50$ and $\sum_{i\neq j}^{10} a_i \cdot a_j = 1100$, then their standard deviation will be

(1) $\sqrt{5}$ (2) $\sqrt{30}$ (3) $\sqrt{15}$ (4) $\sqrt{10}$

Ans. **(1)**

Sol. $(a_1 + a_2 + \ldots + a_{10})^2 = 50^2$

$$\Rightarrow \sum a_i^2 + 2\sum_{i\neq j} a_i a_j = 2500$$

$$\Rightarrow \sum a_i^2 = 300$$

$$\sigma^2 = \frac{\sum a_i^2}{10} - \left(\frac{\sum a_i}{10}\right)^2$$

$$\Rightarrow \sigma^2 = 5 \Rightarrow S.D = \sqrt{5}$$

9. If $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then

Statement-1 : f(–x) is inverse of f(x)

Statement-2 : f(x + y) = f(x)f(y)

(1) Both are true
(2) Both are false
(3) Only statement 1 is true
(4) Only statement 2 is true

Ans. (1)

Sol. $f(x)f(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x-y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

= f(x + y)

$\therefore$ f(x) f(–x) = f(0)

= I

10. If $a = \lim_{x\to 0} \frac{\sqrt{1+\sqrt{1+x^4}} - \sqrt{2}}{x^4}$ and $b = \lim_{x\to 0} \frac{\sin^2 x}{\sqrt{2}-\sqrt{1+\cos x}}$ find $a \cdot b^3$

(1) 16
(2) 32
(3) –16
(4) 48

Ans. (2)

Sol. $a = \lim_{x\to 0} \frac{\sqrt{1+x^4} - 1}{x^4\left[\sqrt{1+\sqrt{1+x^4}} + \sqrt{2}\right]}$

$= \lim_{x\to 0} \frac{x^4}{x^4\left[\sqrt{1+\sqrt{1+x^4}} + \sqrt{2}\right]\left[\sqrt{1+x^4} + 1\right]}$

$= \frac{1}{2\sqrt{2}\times 2} = \frac{1}{4\sqrt{2}}$

$b = \lim_{x\to 0} \frac{\sin^2 x}{(1-\cos x)}\left(\sqrt{2}+\sqrt{1+\cos x}\right)$

$= 2 \times \left(\sqrt{2}+\sqrt{2}\right) = 4\sqrt{2}$

$\therefore ab^3 = \left(4\sqrt{2}\right)^2 = 32$

11. If the minimum distance of centre of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$ from any point on the parabola $y^2 = 4x$ is d, find d^2

Ans. **(20)**

Sol. Normal to parabola is $y = mx - 2m - m^3$

centre $(2, 8) \rightarrow 8 = 2m - 2m - m^3$

$\Rightarrow m = -2$

$\therefore$ p is $(m^2, -2m) = (4, 4)$

$\Rightarrow d^2 = 4 + 16 = 20$

12. If $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 3(\hat{i} - \hat{j} + \hat{k})$, $\vec{a} \times \vec{c} = \vec{b}$ & $\vec{a} \cdot \vec{c} = 3$ find $\vec{a} \cdot (\vec{c} \times \vec{b} - \vec{b} - \vec{c})$

(1) 24 (2) –24 (3) 18 (4) 15

Ans. **(1)**

Sol. $[\vec{a}\ \vec{c}\ \vec{b}] = (\vec{a} \times \vec{c}) \cdot \vec{b} = |\vec{b}|^2 = 27$

$\therefore$ we need $= 27 - 0 - 3 = 24$

13. Consider the line $L : 4x + 5y = 20$. Let two other lines are L_1 and L_2 which trisect the line L and pass through origin, then tangent of angle between lines L_1 and L_2 is

(1) $\frac{20}{41}$ (2) $\frac{30}{41}$ (3) $\frac{40}{41}$ (4) $\frac{10}{41}$

Ans. **(2)**

Sol. Let line L intersect the lines L_1 and L_2 at P and Q

$P\left(\frac{10}{3}, \frac{4}{3}\right), Q\left(\frac{5}{3}, \frac{8}{3}\right)$

$\therefore m_{OA} = \frac{2}{5}$

$m_{OQ} = \frac{8}{5}$

$\tan\theta = \left|\frac{\frac{8}{5} - \frac{2}{5}}{1 + \frac{16}{25}}\right|$

$= \left(\frac{6}{5} \times \frac{25}{41}\right)$

$= \frac{30}{41}$

14. If ${}^{n-1}C_r = (k^2 - 8)\ {}^{n}C_{r+1}$, then the range of 'k' is

(1) $k \in \left(2\sqrt{2}, 3\right]$ (2) $k \in \left(2\sqrt{2}, 3\right)$ (3) $k \in [2, 3)$ (4) $k \in \left(2\sqrt{2}, 8\right)$

Ans. (1)

Sol. ${}^{n-1}C_r = (k^2 - 8)\ \frac{n}{r+1}\ .\ {}^{n-1}C_r$

$\Rightarrow k^2 - 8 = \frac{r+1}{n}$

here $r \in [0, n - 1]$

$\Rightarrow r + 1 \in [1, n]$

$\Rightarrow k^2 - 8 \in \left[\frac{1}{n}, 1\right]$

$\Rightarrow k^2 \in \left[8 + \frac{1}{n}, 9\right]$

$\Rightarrow k \in \left(2\sqrt{2}, 3\right]$

15. If $\alpha x + \beta y + 9\ln|2x + 3y - 8\lambda| = x + C$ is the solution of $(2x + 3y - 2)dx + (4x + 6y - 7)dy = 0$, then $\alpha + \beta + \gamma =$

(1) 18 (2) 19 (3) 20 (4) 21

Ans. **(1)**

Sol. Let $2x + 3y = t$

$\Rightarrow 2 + 3\frac{dy}{dx} = \frac{dt}{dx}$

Now $(t - 2) + (2t - 7)\left(\frac{dt}{dx} - 2\right) \times \frac{1}{3} = 0$

$\Rightarrow -\frac{(3t - 6)}{2t - 7} = \frac{dt}{dx} - 2$

$\Rightarrow \frac{dt}{dx} = \frac{t - 8}{2t - 7}$

$\Rightarrow \int \frac{2t - 7}{t - 8} dt = \int dx$

$\Rightarrow \int 2 + \frac{9}{t - 8} dt = \int dx$

$\Rightarrow 2t + |\ 9\ln|t - 8| = x + C$

$\Rightarrow 2(2x + 3y) + 9\ln|2x + 3y - 8| = x + C$

$\alpha = 4, \beta = 6, \gamma = 8$

16. $f : N - \{1\} \rightarrow N$ and f(n) = highest prime factor of 'n', then f is

(1) one-one, onto (2) many-one, onto

(3) many-one, into (4) one-one, into

Ans. (3)

Sol. '4' is not image of any element $\Rightarrow$ into

$f(10) = 5 = f(15) \Rightarrow$ many-one

17. If P(X) represent the probability of getting a '6' in the X^{th} roll of a die for the first time. Also

$a = P(X = 3)$

$b = P(X \geq 3)$

$c = P\left(\frac{X \geq 6}{x > 3}\right)$, then $\frac{b+c}{a} = ?$

Ans. (12)

Sol. $P(X = 3) = \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} = a$

$P(X \geq 3) = \left(\frac{5}{6}\right)^2 = b$

$P\left(\frac{X \geq 6}{X > 3}\right) = \left(\frac{5}{6}\right)^2 = c$

$\therefore \frac{b+c}{a} = \frac{2\left(\frac{5}{6}\right)^2}{\left(\frac{5}{6}\right)^2 \cdot \frac{1}{6}} = 12$

18. If the angle between two vectors $\vec{a} = \alpha\hat{i} - 4\hat{j} - \hat{k}$ and $\vec{b} = \alpha\hat{i} + \alpha\hat{j} + 4\hat{k}$ is acute then find least positive integral value of α.

(1) 4 (2) 5 (3) 6 (4) 7

Ans. (2)

Sol. $\vec{a} \cdot \vec{b} > 0$

$\Rightarrow \alpha^2 - 4\alpha - 4 > 0$

$\alpha < (2 - 2\sqrt{2})$ or $\alpha > (2 + 2\sqrt{2})$

19. If S = {1, 2, …………… 10} and M = P(S),

If ARB such that $A \cap B \neq \phi$ where $A \in M$, $B \in M$

Then

(1) R is reflexive and symmetric (2) Only symmetric

(3) Only reflexive (4) Symmetric and transitive

Ans. (2)

Sol. $\phi \cap \phi = \phi \quad \Rightarrow (\phi, \phi) \notin R \quad \Rightarrow$ not reflexive.

If $A \cap B \neq \phi \quad \Rightarrow B \cap A \neq \phi \Rightarrow$ Symmetric

If $A \cap B \neq \phi$ and $B \cap C \neq \phi \Rightarrow A \cap C = \phi$

for example A = {1, 2}

B = {2, 3}

C = {3,4}

20. If four points (0, 0), (1, 0), (0, 1), (2k, 3k) are concyclic, then k is

(1) $\frac{4}{13}$ (2) $\frac{5}{13}$ (3) $\frac{7}{13}$ (4) $\frac{9}{13}$

Ans. **(2)**

Sol. Equation of circle **is**

$x(x-1) + y(y-1) = 0$

$x^2 + y^2 - x - y = 0$

B(2k, 3k)

$\Rightarrow 4k^2 + 9k^2 - 2k - 3k = 0$

$\Rightarrow 13k^2 = 5k$

$\Rightarrow k = 0, \frac{5}{13}$

$\therefore k = \frac{5}{13}$

21. If f(x) is differentiable function satisfying $f(x) - f(y) \geq \log \frac{x}{y} + x - y$, then find $\sum_{N=1}^{20} f'\left(\frac{1}{N^2}\right)$

Ans. **(2890)**

Sol. Let x > y

$\lim_{y \to x} \frac{f(x)-f(y)}{x-y} \geq \frac{\log x - \log y}{x-y} + 1$

$f'(x^-) \geq \frac{1}{x} + 1$

Let x < y

$\frac{f(x)-f(y)}{x-y} \leq \frac{\log x - \log y}{x-y} + 1$

$f'(x^+) \leq \frac{1}{x} + 1$

$\Rightarrow f'(x^-) = f'(x^+)$ as f(x) is differentiable function

$f'(x) = \frac{1}{x} + 1$

$f'\left(\frac{1}{N^2}\right) = N^2 + 1$

$$\sum_{N=1}^{20} f'\left(\frac{1}{N^2}\right) = \sum (N^2+1) = \frac{20 \times 21 \times 41}{6} + 20 = 2890$$

22. Let $\frac{dx}{dt} + ax = 0$ and $\frac{dy}{dt} + by = 0$ where y(0) = 1, x(0) = 2, and x(t) = y(t), then t is

(1) $\frac{\ln 3}{a-b}$ (2) $\frac{\ln 2}{b-a}$ (3) $\frac{\ln 2}{a-b}$ (4) $\frac{\ln 3}{b-a}$

Ans. **(3)**

Sol. $\frac{dx}{dt} + ax = 0$

$\Rightarrow \ln x = -at + c$

$x(0) = 2 \Rightarrow c = \ln 2$

$\therefore x = 2e^{-at}$

$\frac{dy}{dt} + by = 0 \Rightarrow y = e^{-bt}$

$x(t) = g(t)$

$2e^{-at} = e^{-bt}$

$\Rightarrow t = \frac{\ln 2}{a-b}$

23. If H(a, b) is the orthocentre of ΔABC where A(1, 2), B(2,3) & C(3, 1), then find $\frac{36I_1}{I_2}$ if

$I_1 = \int_a^b x\sin(4x - x^2)\,dx$ and $I_2 = \int_a^b \sin(4x - x^2)\,dx$

Ans. **(72)**

Sol. ΔABC is isosceles

$\Rightarrow$ H lies on angle bisector passing through (3, 1) which is x + y = 4

$\therefore a + b = 4$

Now apply (a + b – x) in I_1

$2I_1 = \int_a^b 4\sin(4x - x^2)\,dx$

$\Rightarrow 2I_1 = 4I_2$

$\Rightarrow \frac{I_1}{I_2} = 2$

$\therefore \frac{36I_1}{I_2} = 72$

24. $f(x) = \begin{cases} 2^{\frac{\sin(x-3)}{x-[x]}} & , \quad x > 3 \\ -\dfrac{a(x^2-7x+12)}{b|x^2-7x+12|} & , \quad x < 3 \\ b & , \quad x = 3 \end{cases}$. Find number of ordered pairs (a, b) so that f(x) is continuous

at x = 3

Ans. **(1)**

Sol. LHL = RHL = f(3)

$-\dfrac{a}{b} = 2^1 = b$

$\Rightarrow$ b = 2 and a = – 4

$\Rightarrow$ (a ,b) = (–4,2)

25. Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}$, B = [$B_1$ B_2 B_3] where B_1, B_2, B_3 are column matrices such that

$AB_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $AB_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $AB_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

α = sum of diagonal elements of B

$\beta = |B|$, then find $|\alpha^3 + \beta^3|$

Ans. **(1.125)**

Sol. $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{1}{2} \\ 1 & -2 & 0 \end{bmatrix}$

$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 2 \end{bmatrix}$, $B_3 = \begin{bmatrix} 2 \\ -\frac{5}{2} \\ -1 \end{bmatrix}$

$\text{Tr}(B) = -\dfrac{1}{2}$

$|B| = -1$

$\therefore\ a = -\dfrac{1}{2}$, $b = -1$

$|\alpha^3 + \beta^3| = \dfrac{9}{8} = 1.125$

26. If cos(2x) – a sinx = 2a – 7 has a solution for $a \in [p, q]$ and r = tan9° + tan63° + tan81° + tan27°, then p.q. r = ?

(1) $40\sqrt{5}$ (2) $32\sqrt{5}$ (3) $30\sqrt{5}$ (4) $48\sqrt{5}$

Ans. **(4)**

Sol. $2(\sin^2 x - 4) + a(\sin x + 2) = 0$

$2(\sin x - 2) + a = 0$

$\Rightarrow a = 4 - 2\sin x$

$a \in [2, 6]$

Also, $r = \left(\tan 9^\circ + \dfrac{1}{\tan 9^\circ}\right) + \left(\tan 27^\circ + 1\dfrac{1}{\tan 27^\circ}\right)$

$= \dfrac{2}{\sin 18^\circ} + \dfrac{2}{\sin 54^\circ}$

$= \dfrac{2\times 4}{\sqrt{5}-1} + \dfrac{2\times 4}{\sqrt{5}+1}$

$= \dfrac{8\times 2\sqrt{5}}{4} = 4\sqrt{5}$

$\therefore$ pqr $= 48\sqrt{5}$